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The general form of the star–product on the Grassman algebra

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Abstract

We study the general form of the noncommutative associative product (the star–product) on the Grassman algebra; the star–product is treated as a deformation of the usual "pointwise" product. We show that up to a similarity transformation, there is only one such product. The relation of the algebra $\mathcal{F}$, the algebra of elements of the Grassman algebra with the star–product as a product, to the Clifford algebra is discussed.

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1 Introduction

The noncommutative associative product of functions on a phase space (the star–product, the $*$–product in what follows) arises frequently in the physical literature. The traditional region where the $*$–product construction is conventionally used is the problem on quantizing the classical theory (i.e., the Poisson bracket) in the case where the phase space is a nontrivial manifold, the so–called geometrical or deformation quantizations (see [1] and references wherein). The relations of the $*$–product to quantum groups were recently found [2], [3]. The gauge theories on the noncommutative spaces, the so–called noncommutative gauge theories, are formulated in terms of the $*$–product (see [4] and [5] and references wherein).

The structure and the properties of the $*$–product on the usual (even) manifolds are investigated in details [1], [6] and [7]. The BRST formulation of the Fedosov construction [9] for the $*$–product on symplectic manifolds was recently found [8]. On the other hand, the $*$–product on supermanifolds is not sufficiently studied. It is easy to extend formally the noncommutative product proposed in [10] to the supercase [11] (see also [12], [13] [14]), however, a discussion of the uniqueness of the $*$–product (the uniqueness of the "pointwise" product deformation) is absent in the literature. In the present paper, we investigate the general form of the associative $*$–product, treated as a deformation of the "pointwise" product, on the Grassman algebra of a finite number of generators. We show that up to a similarity transformation, only one such product exists.

The paper is organized as follows. In Sec. 2, we describe the formulation of the problem. In Sec. 3, we present the solution of the associativity equation for the $*$–product. In Sec. 4, we prove the uniqueness (up to a similarity transformation) of the $*$–product and formulate the basic results. In Sec. 5, the relation of the algebra $\mathcal{F}$ of elements of the Grassman algebra with $*$–product as a product to the Clifford algebra is discussed.

Notation and conventions.

ξ^α, $\alpha = 1, \ldots, n$ are odd anticommuting generators of the Grassman algebra:

$$\varepsilon(\xi^\alpha) = 1, \quad \xi^\alpha\xi^\beta + \xi^\beta\xi^\alpha = 0, \quad \partial_\alpha \equiv \frac{\partial}{\partial\xi^\alpha},$$

$\varepsilon(A)$ denotes the Grassman parity of A;

$$[\xi^\alpha]^0 \equiv 1, \quad [\xi^\alpha]^k \equiv \xi^{\alpha_1}\cdots\xi^{\alpha_k}, 1 \le k \le n, \quad [\xi^\alpha]^k = 0, k > n,$$

$$[\partial_\alpha]^0 \equiv 1, \quad [\partial_\alpha]^k \equiv \partial_{\alpha_1}\cdots\partial_{\alpha_k}, 1 \le k \le n, \quad [\partial_\alpha]^k = 0, k > n,$$

$$[\overleftarrow{\partial}_\alpha]^k \equiv \overleftarrow{\partial}_{\alpha_1}\cdots\overleftarrow{\partial}_{\alpha_k},$$

$$T_{\ldots[\alpha]_k\ldots} \equiv T_{\ldots\alpha_1\ldots\alpha_k\ldots}, \quad T_{\ldots\alpha_i\alpha_{i+1}\ldots} = -T_{\ldots\alpha_{i+1}\alpha_i\ldots}, \quad i = 1, \ldots, k-1.$$

2 Setting the problem

We consider the Grassman algebra $\mathcal{G}_K$ over the field $K = \mathbf{C}$ or $\mathbf{R}$ with the generators ξ^α, $\alpha = 1, 2, \ldots, n$. The general element f of the algebra (a function of the generators) is

$$f \equiv f(\xi) = \sum_{k=0}^{n} \frac{1}{k!} f_{[\alpha]_k}[\xi^\alpha]^k, \quad f_{[\alpha]_0} \equiv f_0, f_{[\alpha]_k} \in K, \quad \forall k.$$

We also introduce the basis $\{e^I\}$ in the algebra $\mathcal{G}_K$:

$$\begin{array}{c} I = 0,\ \alpha_i,\ \alpha_{i_1}\alpha_{i_2},\ \ldots,\ \alpha_{i_1}\cdots\alpha_{i_k},\ \ldots,\quad i_1 < i_2 < \cdots < i_k, \\ e^0 = 1,\quad e^{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_k}} = \xi^{\alpha_{i_1}}\xi^{\alpha_{i_2}}\cdots\xi^{\alpha_{i_k}},\quad k = 0, 1, \ldots, n. \end{array} \tag{1}$$

The basis contains 2^n elements. Any element f of the algebra $\mathcal{G}_K$ can be uniquely represented as

$$f = \sum_I f_I e^I,\quad f_I \in K. \tag{2}$$

The integral over the generators is normalized as follows:

$$\int d\xi[\xi^\alpha]^k = 0,\ k \le n-1,\quad \int d\xi[\xi^\alpha]^n = \varepsilon^{[\alpha]_n},\quad \varepsilon(d\xi) = n\,(mod\ 2).$$

$$\int d\xi'\delta(\xi - \xi')f(\xi') = f(\xi),\quad \varepsilon(\delta(\xi)) = n\,(mod\ 2).$$

We are interested in the general form of the associative even $*$–product for arbitrary elements f_1, f_2:

$$f_3(\xi) = f_1 * f_2(\xi) \equiv P(\xi|f_1, f_2),$$

i.e., the general form of the bilinear mapping $\mathcal{G}_K \times \mathcal{G}_K \to \mathcal{G}_K$ satisfying the associativity equation:

$$f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3, \tag{3}$$

and having the parity property

$$\varepsilon(f_1 * f_2) = \varepsilon(f_1) + \varepsilon(f_2)\,(mod\ 2).$$

We solve associativity equation (3) in terms of the series in the deformation parameter $\hbar$:

$$* = \sum_{k=0} \hbar^k *_k, \tag{4}$$

where the $*_0$–product, that is the boundary condition, is the usual "pointwise" product:

$$f_1 *_0 f_2(\xi) = f_1(\xi)f_2(\xi). \tag{5}$$

Thus, we search for the general form of a possible even associative deformation of the "pointwise" product on the Grassman algebra.

It is easy to verify that the $*$–product (as any bilinear mapping) can be written in the coordinate representation form:

$$f_3(\xi) = \int d\xi_2 d\xi_1 P(\xi|\xi_1, \xi_2) f_1(\xi_1) f_2(\xi_2),\quad \varepsilon(P(\xi|\xi_1, \xi_2)) = 0 \tag{6}$$

with a certain function (the kernel) $P(\xi|\xi_1, \xi_2)$, or in the momentum representation form:

$$f_3(\xi) = f_1(\xi) \sum_{k_1,k_2=0}^{n} [\overleftarrow{\partial}_\beta]^{k_1} P^{[\beta]_{k_1}|[\gamma]_{k_2}}(\xi)[\partial_\gamma]^{k_2} f_2(\xi),\quad \varepsilon(P^{[\beta]_{k_1}|[\gamma]_{k_2}}(\xi)) = k_1 + k_2\,(mod\ 2)$$

with certain coefficient functions $P^{[\beta]_{k_1}|[\gamma]_{k_2}}(\xi)$.

Let some associative even $*$–product exist, and let T be the nonsingular even linear mapping $\mathcal{G}_K \to \mathcal{G}_K$:

$$Tf(\xi) \equiv T(\xi|f) = \int d\xi_1 T(\xi|\xi_1)f(\xi_1) = \sum_{k=0}^{n} T^{[\alpha]_k}(\xi)[\partial_\alpha]^k f(\xi),$$

$$\varepsilon(T(\xi|\xi_1)) = n\,(mod\ 2), \quad \varepsilon(T^{[\alpha]_k}(\xi)) = k\,(mod\ 2).$$

Then the $*_T$–product

$$f_1 *_T f_2(\xi) \equiv T^{-1}\Big(Tf_1 * Tf_2\Big)(\xi)$$

is also an associative even $*$–product. We say that the $*_T$–product and the $*$–product are related by a (T–)similarity transformation, or (T–)equivalent. We note that any nonsingular even change of the algebra generators:

$$T_\eta \xi^\alpha = \eta^\alpha(\xi),\ T_\eta^{-1}\xi^\alpha = T_\zeta \xi^\alpha = \zeta^\alpha(\xi), \quad T_\eta f(\xi) = f(\eta(\xi)),\ T_\eta^{-1} f(\xi) = T_\zeta f(\xi) = f(\zeta(\xi)), \quad (7)$$

where $\zeta^\alpha(\xi)$ is the inverse change, $\eta^\alpha(\zeta(\xi)) = \xi^\alpha$, $\varepsilon(\eta^\alpha) = \varepsilon(\zeta^\alpha) = 1$, induces the similarity transformation. Any nonsingular even change can be represented as

$$T = e^{u^\alpha(\xi)\partial_\alpha}$$

with some functions $u^\alpha(\xi)$, $\varepsilon(u^\alpha(\xi)) = 1$.

The associativity equation for the first order deformation $*_1$ is:

$$\Big(f_1 *_1 (f_2 f_3)\Big)(\xi) + f_1(\xi)\Big(f_2 *_1 f_3\Big)(\xi) - \Big(f_1 *_1 f_2\Big)(\xi) f_3(\xi) - \Big((f_1 f_2) *_1 f_3\Big)(\xi) = 0. \quad (8)$$

The trivial solution of eq. (8) is $*_{1triv}$ that is the first order approximation to the $*_{0T}$–product T–equivalent to the usual product with $T = I + \hbar t_1 + O(\hbar^2)$, I denotes the identical mapping. This solution is of the form

$$f_1 *_{1triv} f_2 = (t_1 f_1) f_2 + f_1(t_1 f_2) - t_1(f_1 f_2). \quad (9)$$

The problem is to describe all nontrivial solutions of eq. (8), which appears to be sufficient for describing all solutions of associativity equation (3).

Consider the algebra $\mathcal{A} = \sum_{\oplus} \mathcal{A}_k$ of even k–linear functionals $\Phi_k(\xi|f_1, \dots, f_k) \in \mathcal{A}_k$ (of k–linear mappings $(\mathcal{G}_K \times)^k \to \mathcal{G}_K$), $\varepsilon(\Phi_k(\xi|f_1, \dots, f_k)) = \varepsilon(f_1) + \dots + \varepsilon(f_k)$ $(mod\ 2)$, $k \geq 1$, $\mathcal{A}_0 = K$, with the (noncommutative) product (the mapping $\mathcal{A}_k \times \mathcal{A}_p \to \mathcal{A}_{k+p}$)

$$\Phi_k \Psi_p(\xi|f_1, \dots, f_{k+p}) = \Phi_k(\xi|f_1, \dots, f_k)\Psi_p(\xi|f_{k+1}, \dots, f_{k+p}).$$

In this algebra, there exists the natural grading g, $g(\Phi_k) = k$, turning the algebra $\mathcal{A}$ into the graded algebra, $g(\mathcal{A}_k) = k$, and the linear operator d_H, the differential coboundary Hochschild operator $\mathcal{A}_k \to \mathcal{A}_{k+1}$, $g(d_H) = 1$, acting according to the rule:

$$d_H \Phi_k(\xi|f_1, \dots, f_{k+1}) = f_1(\xi)\Phi_k(\xi|f_2, \dots, f_{k+1}) +$$

$$+\sum_{i=1}^{k}(-1)^i\Phi_k(\xi|f_1,\ldots,f_{i-1},f_if_{i+1},f_{i+2},\ldots,f_{k+1})+(-1)^{k+1}\Phi_k(\xi|f_1,\ldots,f_k)f_{k+1}(\xi),$$

$$d_H\mathcal{A}_0=0.$$

It is easy to verify that the operator d_H is a graded odd differentiation,

$$d_H(\Phi_k\Psi_p)=(d_H\Phi_k)\Psi_p+(-1)^{g(\Phi_k)}\Phi_k d_H\Psi_p,$$

and nilpotent,

$$d_H^2=0.$$

Equations (8) and (9) are written in terms of the operator d_H as

$$d_H*_1=0, \tag{10}$$

$$*_{1triv}=d_Ht_1. \tag{11}$$

This means that $*_1$ belongs to the second Hochschild cohomology group, while the first order trivial deformations of the "pointwise" product are coboundaries, i.e., they belong to the zero Hochschild cohomology.

3 Solution of the associativity equation

We find the general solution of eq. (8) for the first order in $\hbar$ deformation of the "pointwise" product. We follow the method in [15]. It is convenient to use coordinate representation (6). Associativity equation (3) for the $*$–product is reduced to the equation for the kernel:

$$\int d\xi_4P(\xi|\xi_4,\xi_3)P(\xi_4|\xi_1,\xi_2)=(-1)^n\int d\xi_4P(\xi|\xi_1,\xi_4)P(\xi_4|\xi_2,\xi_3). \tag{12}$$

With the boundary condition taken into account, the kernel of the $*$–product has the following power series expansion in $\hbar$:

$$P(\xi|\xi_1,\xi_2)=\delta(\xi-\xi_1)\delta(\xi-\xi_2)+\hbar m_1(\xi|\xi_1,\xi_2)+O(\hbar^2).$$

The equation for the kernel m_1 follows from eq. (12):

$$\delta(\xi_1)\tilde{m}_1(\xi|\xi_2,\xi_3)+\delta(\xi_3-\xi_2)\tilde{m}_1(\xi|\xi_1,\xi_3)=\delta(\xi_3)\tilde{m}_1(\xi|\xi_1,\xi_2)+\delta(\xi_1-\xi_2)\tilde{m}_1(\xi|\xi_1,\xi_3), \tag{13}$$

$$\tilde{m}_1(\xi|\xi_1,\xi_2)\equiv m_1(\xi|\xi_1+\xi,\xi_2+\xi).$$

Multiplying eq. (13) by ξ_1^α and integrating over $d\xi_1$, we obtain

$$\xi_1^\alpha\tilde{m}_1(\xi|\xi_1,\xi_2)=[(-1)^n\delta(\xi_1-\xi_2)-\delta(\xi_2)]a_1^\alpha(\xi|\xi_1), \tag{14}$$

$$a_1^\alpha(\xi|\xi_1)=\int d\xi_2\xi_2^\alpha\tilde{m}_1(\xi|\xi_2,\xi_1),\quad \varepsilon(a_1^\alpha)=n+1\,(mod\ 2).$$

Multiplying relation (14) by ξ_1^β and taking the antisymmetry in α and β of the l.h.s. of the resulting relation into account, we find

$$[(-1)^n\delta(\xi_1-\xi_2)-\delta(\xi_2)](\xi_1^\alpha a_1^\beta(\xi|\xi_1)+\xi_1^\beta a_1^\alpha(\xi|\xi_1))=0,$$

whence it follows
$$\xi_1^\alpha a_1^\beta(\xi|\xi_1) + \xi_1^\beta a_1^\alpha(\xi|\xi_1) = -2\delta(\xi_1)\omega_1^{\alpha\beta}(\xi), \tag{15}$$
with a certain function $\omega_1^{\alpha\beta}(\xi)$, $\omega_1^{\alpha\beta} = \omega_1^{\beta\alpha}$, $\varepsilon(\omega_1^{\alpha\beta}) = 0$. We represent the function $a_1^\alpha(\xi|\xi_1)$ as
$$a_1^\alpha(\xi|\xi_1) = -\omega_1^{\alpha\beta}(\xi)\frac{\partial}{\partial\xi_1^\beta}\delta(\xi_1) + \bar{a}_1^\alpha(\xi|\xi_1),$$
It follows from eq. (15) that
$$\xi_1^\alpha \bar{a}_1^\beta(\xi|\xi_1) + \xi_1^\beta \bar{a}_1^\alpha(\xi|\xi_1) = 0,$$
from where we find
$$\bar{a}_1^\alpha(\xi|\xi_1) = \xi_1^\alpha \bar{a}_1(\xi|\xi_1), \quad \varepsilon(\bar{a}_1) = n\,(mod\ 2).$$
Substituting the obtained representation for a_1^α into relation (14), we obtain:
$$\xi_1^\alpha \tilde{m}_1(\xi|\xi_1,\xi_2) = \xi_1^\alpha[\delta(\xi_1-\xi_2)\bar{a}_1(\xi|\xi_1) - (-1)^n\delta(\xi_2)\bar{a}_1(\xi|\xi_1) + \delta(\xi_1)\frac{\overleftarrow{\partial}}{\partial\xi_1^\beta}\omega_1^{\beta\gamma}(\xi)\frac{\partial}{\partial\xi_2^\gamma}\delta(\xi_2)],$$
whence it follows
$$\tilde{m}_1(\xi|\xi_1,\xi_2) = \delta(\xi_1-\xi_2)\bar{a}_1(\xi|\xi_1) - (-1)^n\delta(\xi_2)\bar{a}_1(\xi|\xi_1) + \delta(\xi_1)\bar{b}_1(\xi|\xi_2) + \delta(\xi_1)\frac{\overleftarrow{\partial}}{\partial\xi_1^\beta}\omega_1^{\beta\gamma}(\xi)\frac{\partial}{\partial\xi_2^\gamma}\delta(\xi_2)],$$
with a certain function $\bar{b}_1(\xi|\xi_2)$, $\varepsilon(\bar{b}_1) = n\,(mod\ 2)$.

Thus, the first order deformation of the "pointwise" product can be represented in the form
$$f_1 *_1 f_2(\xi) = m_1(\xi|f_1,f_2) =$$
$$f_1(\xi)\overleftarrow{\partial}_\alpha\omega_1^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) + f_1(\xi)t_1(\xi|f_2) - t_1(\xi|f_1f_2) + t_1(\xi|f_1)f_2(\xi) + c_1(\xi|f_1f_2) - c_1(\xi|f_1)f_2(\xi),$$
where we introduce the notation $t_1(\xi|\xi_1) = \bar{b}_1(\xi|\xi_1-\xi)$, $c_1(\xi|\xi_1) = \bar{a}_1(\xi|\xi_1-\xi) + \bar{b}_1(\xi|\xi_1-\xi)$, $t_1(\xi|f) \equiv \int d\xi_1 t_1(\xi|\xi_1)f(\xi_1)$, $c_1(\xi|f) \equiv \int d\xi_1 c_1(\xi|\xi_1)f(\xi_1)$. Substituting this representation for the $*_1$–product into eq. (8) (or in eq. (13)), we find
$$f_1(\xi)[c_1(\xi|f_2f_3) - c_1(\xi|f_2)f_3(\xi)] = 0,$$
or
$$c_1(\xi|f_1f_2) - c_1(\xi|f_1)f_2(\xi) = 0.$$
The expression $f_1(\xi)\overleftarrow{\partial}_\alpha\omega_1^{\alpha\beta}(\xi)\partial_\beta f_2(\xi)$ can not be represented as $d_H t(\xi|f_1,f_2)$. To prove this, we use the momentum representation for the functional $t(\xi|f)$:
$$t(\xi|f) = \sum_{k=0} t^{[\alpha]_k}(\xi)[\partial_\alpha]^k f(\xi), \quad \varepsilon(t^{[\alpha]_k}) = k\,(mod\ 2).$$
The summand in $d_H t(\xi|f_1,f_2)$ with total order of the derivatives of the functions $f_1(\xi)$ and $f_2(\xi)$ equal to 2 can arise only from the summand in $t(\xi|f)$ with the coefficient $t^{[\alpha]_2}(\xi)$. Choosing
$$t(\xi|f) = \frac{1}{2}t^{\alpha\beta}(\xi)\partial_\alpha\partial_\beta f(\xi), \quad t^{\alpha\beta} = -t^{\beta\alpha}, \quad \varepsilon(t^{\alpha\beta}) = 0,$$

we obtain that the equality must be true

$$f_1(\xi)\overleftarrow{\partial}_\alpha\omega_1^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) = d_H t(\xi|f_1,f_2) = f_1(\xi)\overleftarrow{\partial}_\alpha t^{\alpha\beta}(\xi)\partial_\beta f_2(\xi),$$

or

$$\omega_1^{\alpha\beta}(\xi) = t^{\alpha\beta}(\xi).$$

The last equality is possible only for $\omega_1^{\alpha\beta}(\xi)=0$ because $\omega_1^{\alpha\beta}(\xi)$ is a symmetric matrix while $t^{\alpha\beta}(\xi)$ is an antisymmetric one.

We have thus found that the general solution of eq. (8) has the form

$$f_1 *_1 f_2(\xi) = m_1(\xi|f_1,f_2) = f_1(\xi)\overleftarrow{\partial}_\alpha\omega_1^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) + f_1(\xi)t_1(\xi|f_2) - t_1(\xi|f_1f_2) + t_1(\xi|f_1)f_2(\xi) =$$

$$= f_1(\xi)\overleftarrow{\partial}_\alpha\omega_1^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) + d_H t_1(\xi|f_1,f_2).$$

If we perform the similarity transformation T of the $*$–product,

$$f_1 * f_2 \quad\longrightarrow\quad f_1 *_{T_1} f_2 = T_1^{-1}(T_1 f_1 * T_1 f_2),$$

with the operator[1]

$$T_1 = e^{-\hbar\hat{t}_1},$$

then it is easy to verify that P_{T_1} has the following power series expansion in $\hbar$:

$$P_{T_1}(\xi|f_1,f_2) = f_1(\xi)f_2(\xi) + \hbar m_1(\xi|f_1,f_2) - \hbar d_H t_1(\xi|f_1,f_2) + \hbar^2 m_{T_12}(\xi|f_1,f_2) + O(\hbar^3) =$$

$$= f_1(\xi)f_2(\xi) + \hbar f_1(\xi)\overleftarrow{\partial}_\alpha\omega_1^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) + \hbar^2 m_{T_12}(\xi|f_1,f_2) + O(\hbar^3). \qquad (16)$$

In what follows, we omit the index "T_1" at the $*_{T_1}$–product and at the kernels P_{T_1} and m_{T_12}.

It is now useful to introduce the $*$–commutator $[f_1,f_2]_*(\xi)$ for two functions $f_1(\xi)$ and $f_2(\xi)$ defined by

$$[f_1,f_2]_*(\xi) \equiv \frac{1}{2\hbar}\Big(f_1 * f_2(\xi) - (-1)^{\varepsilon(f_1)\varepsilon(f_2)} f_2 * f_1(\xi)\Big)$$

and to define the Jacobian $J(\xi|f_1,f_2,f_3)$ of three functions $f_1(\xi)$, $f_2(\xi)$ and $f_3(\xi)$ as

$$J(\xi|f_1,f_2,f_3) \equiv (-1)^{\varepsilon(f_1)\varepsilon(f_3)}[[f_1,f_2]_* f_3]_*(\xi) + \text{cycle}\,(1,2,3). \qquad (17)$$

We have

$$J(\xi|f_1,f_2,f_3) = 0 \qquad (18)$$

as the consequence of associativity equation (3) (the $*$–commutator satisfies the Jacobi identity).

The direct calculation of the Jacobian $J(\xi|f_1,f_2,f_3)$ for $*$–product (16) gives

$$J(\xi|f_1,f_2,f_3) = -(-1)^{\varepsilon(f_2)+\varepsilon(f_1)\varepsilon(f_3)}\Omega_1^{\alpha\beta\gamma}(\xi)\partial_\alpha f_1(\xi)\partial_\beta f_2(\xi)\partial_\gamma f_3(\xi) + O(\hbar),$$

[1] We note, that the similarity transformation T of the "pointwise" product $*_0$ with $T = 1 + \hbar t_1 + 0(\hbar^2)$ is of the form

$$*_{0T} = *_0 + d_H t_1 + O(\hbar^2).$$

$$\Omega_1^{\alpha\beta\gamma}(\xi) = \omega_1^{\alpha\delta}(\xi)\partial_\delta\omega_1^{\beta\gamma}(\xi) + \omega_1^{\gamma\delta}(\xi)\partial_\delta\omega_1^{\alpha\beta}(\xi) + \omega_1^{\beta\delta}(\xi)\partial_\delta\omega_1^{\gamma\alpha}(\xi).$$

It follows from eq. (18) that the function $\omega_1^{\alpha\beta}(\xi)$ satisfies the Jacobi identity

$$\omega_1^{\alpha\delta}(\xi)\partial_\delta\omega_1^{\beta\gamma}(\xi) + \omega_1^{\gamma\delta}(\xi)\partial_\delta\omega_1^{\alpha\beta}(\xi) + \omega_1^{\beta\delta}(\xi)\partial_\delta\omega_1^{\gamma\alpha}(\xi) = 0.$$

We note that the function $\omega^{\alpha\beta}(\xi)$ transforms as a tensor under a change of the generators $\xi^\alpha \to \eta^\alpha(\xi)$. This means the following. Let operator T_η (see eq. (7)) describes a change of the generators. The $\hbar$ power series expansion of the $*$–product $T_\eta^{-1}P(\xi|T_\eta f_1, T_\eta f_2)$, where $P(\xi|f_1, f_2)$ is given by formula (16), is

$$T_\eta^{-1}P(\xi|T_\eta f_1, T_\eta f_2) = f_1(\xi)f_2(\xi) + \hbar f_1(\xi)\overleftarrow{\partial}_\alpha\omega_1'^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) + O(\hbar^2),$$
$$\omega_1'^{\alpha\beta}(\xi) = \xi^\alpha\frac{\overleftarrow{\partial}}{\partial\zeta^\gamma}\omega_1^{\gamma\delta}(\zeta)\frac{\partial}{\partial\zeta^\delta}\xi^\beta. \tag{19}$$

Therefore, the tensor function $\omega_1^{\alpha\beta}(\xi)$ is the symplectic metric of the Poisson bracket. We assume that the metric $\omega_1^{\alpha\beta}$ is nonsingular. In this case, there exists a change of the generators $\xi^\alpha \to \eta^\alpha(\xi)$ that reduces the symplectic metric to the canonical form

$$\omega_1'^{\alpha\beta}(\xi) = \lambda_\alpha\delta_{\alpha\beta}, \tag{20}$$

where $\lambda_\alpha = 1$ in the case of the Grassman algebra over complex numbers ($\mathcal{G}_{\mathbf{C}}$) and $\lambda_\alpha = \pm 1$ in the case of the Grassman algebra over real numbers ($\mathcal{G}_{\mathbf{R}}$). We assume that the appropriate similarity transformation is performed such that the tensor function in formula (16) is of canonical form (20).

We represent the $*$–product as

$$P(\xi|f_1, f_2) = P_G^{\hbar\lambda}(\xi|f_1, f_2) + \hbar^2 m_2'(\xi|f_1, f_2) + O(\hbar^3), \tag{21}$$

where

$$P_G^{\hbar\lambda}(\xi|f_1, f_2) \equiv f_1 *_{G(\hbar\lambda)} f_2(\xi) = f_1(\xi)e^{\hbar Q_\lambda}f_2(\xi) = f_1(\xi)f_2(\xi) + \hbar f_1(\xi)\overleftarrow{\partial}_\alpha\lambda_\alpha\partial_\alpha f_2(\xi) + O(\hbar^2),$$
$$Q_\lambda = \sum_\alpha \lambda_\alpha\overleftarrow{\partial}_\alpha\partial_\alpha. \tag{22}$$

$P_G^{\hbar\lambda}(\xi|f_1, f_2)$ itself satisfies the associativity equation (see Appendix). Substituting expansion (21) into eq. (12), we find that the kernel m_2' satisfies just eq. (13), whence it follows

$$m_2'(\xi|f_1, f_2) = f_1(\xi)\overleftarrow{\partial}_\alpha\omega_2^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) + d_H t_2(\xi|f_1, f_2)$$

with a certain function $\omega_2^{\alpha\beta}(\xi) = \omega_2^{\beta\alpha}(\xi)$, $\varepsilon(\omega_2^{\alpha\beta}) = 0$, and a kernel $t_2(\xi|\xi_1)$, $\varepsilon(t_2) = n\,(mod\ 2)$. The summand $d_H t_2(\xi|f_1, f_2)$ can be transformed out by the similarity transformation with the operator $T = \exp(-\hbar^2\hat{t}_2)$. Assuming that this similarity transformation has been already performed, we obtain that the $*$–product can be represented as

$$P(\xi|f_1, f_2) = P_G^{\hbar\lambda}(\xi|f_1, f_2) + \hbar^2 f_1(\xi)\overleftarrow{\partial}_\alpha\omega_2^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) + \hbar^3 m_3'(\xi|f_1, f_2) + O(\hbar^4). \tag{23}$$

We assume that the $*$–product can be represented as

$$P(\xi|f_1, f_2) = P_G^{\hbar\lambda}(\xi|f_1, f_2) + \hbar^k f_1(\xi)\overleftarrow{\partial}_\alpha\omega_k^{\alpha\beta}(\xi)\partial_\beta f_2(\xi) + \hbar^{k+1}m_{k+1}(\xi|f_1, f_2) + O(\hbar^{k+2}), \tag{24}$$

$$\omega_k^{\alpha\beta} = \omega_k^{\beta\alpha}, \quad \varepsilon(\omega_k^{\alpha\beta}) = 0, \quad k \geq 2,$$

after the appropriate similarity transformation.

Then the direct calculation of the Jacobian $J(\xi|f_1, f_2, f_3)$ for this $*$–product (24) gives

$$J(\xi|f_1, f_2, f_3) = -(-1)^{\varepsilon(f_2)+\varepsilon(f_1)\varepsilon(f_3)}\hbar^{k-1}\Omega_k^{\alpha\beta\gamma}(\xi)\partial_\alpha f_1(\xi)\partial_\beta f_2(\xi)\partial_\gamma f_3(\xi) + O(\hbar^k),$$

$$\Omega_k^{\alpha\beta\gamma}(\xi) = \lambda_\alpha\partial_\alpha\omega_k^{\beta\gamma}(\xi) + \lambda_\gamma\partial_\gamma\omega_k^{\alpha\beta}(\xi) + \lambda_\beta\partial_\beta\omega_k^{\gamma\alpha}(\xi).$$

It follows from eq. (18) that the function $\omega_k^{\alpha\beta}(\xi)$ satisfies the relation (the Bianchi identity)

$$\lambda_\alpha\partial_\alpha\omega_k^{\beta\gamma}(\xi) + \lambda_\gamma\partial_\gamma\omega_k^{\alpha\beta}(\xi) + \lambda_\beta\partial_\beta\omega_k^{\gamma\alpha}(\xi) = 0,$$

whence we find

$$\omega_k^{\alpha\beta}(\xi) = \lambda_\alpha\partial_\alpha\omega_k^{\beta}(\xi) + \lambda_\beta\partial_\beta\omega_k^{\alpha}(\xi), \quad \varepsilon(\omega_k^{\alpha}) = 1.$$

We perform the similarity transformation of $*$–product (24) with the operator $T = \exp\left(-\hbar^{k-1}\omega_k^{\alpha}(\xi)\partial_\alpha\right)$. Taking the relations

$$e^{-\hat{t}}\left(e^{\hat{t}}f_1(\xi)e^{\hat{t}}f_2(\xi)\right) = f_1(\xi)f_2(\xi), \quad \hat{t} = \omega^\alpha(\xi)\partial_\alpha, \quad \varepsilon(\omega^\alpha) = 1,$$

$$e^{-\hbar^k\hat{t}}P_G^{\hbar\lambda}(\xi|e^{\hbar^k\hat{t}}f_1, e^{\hbar^k\hat{t}}f_2) = P_G^{\hbar\lambda}(\xi|f_1, f_2) +$$
$$+\hbar^{k+1}f_1(\xi)\overleftarrow{\partial}_\alpha\left(\lambda_\alpha\partial_\alpha\omega^\beta(\xi) + \lambda_\beta\partial_\beta\omega^\alpha(\xi)\right)\partial_\beta f_2(\xi) + O(\hbar^{k+2}), \quad k \geq 1,$$

into account, we obtain that the transformed $*$–product has the form (index T is omitted)

$$P(\xi|f_1, f_2) = P_G^{\hbar\lambda}(\xi|f_1, f_2) + \hbar^{k+1}m'_{k+1}(\xi|f_1, f_2) + O(\hbar^{k+2}),$$

and m'_{k+1} satisfies eq. (13). It remains to apply the induction method.

We have thus shown that any associative $*$–product on the Grassman algebra with boundary condition (5) can be obtained from the $*_{G(\hbar\lambda)}$–product by a similarity transformation.

4 Uniqueness

We have established that any $*$–product with boundary condition (5) and a nonsingular metric in the first–order deformation can be reduced to the form of the $*_{G(\hbar\lambda)}$–product by the similarity transformations generated by the operators of the form

$$T = T_\eta(1 + \hbar t_1 + O(\hbar^2)), \tag{25}$$

T_η is the operator of a nonsingular change of the generators. It is essential that it is impossible to reduce the $*$–product with a nonsingular tensor $\omega_1^{\alpha\beta}(\xi)$ to the "pointwise" product by a similarity transformations of this type because t_1 does not contribute to $\omega_1^{\alpha\beta}(\xi)$ and T_η does not violate the nonsingularity of the tensor $\omega_1^{\alpha\beta}(\xi)$ (see eq. (19)). The question arises whether there exists a similarity transformation generated by an operators T of the more general form that does reduce the $*_{G(\hbar\lambda)}$–product to the "pointwise" one? We now show that the answer to this question is in negative.

In fact, any nonsingular operator T can be written as

$$T = T'(1 + O(\hbar)),$$

with a certain nonsingular operator $T' = T|_{\hbar=0}$. By virtue of the boundary condition for the $*$–product (see eqs. (4) and (5)) the T'–similarity transformation should leave the "pointwise" product invariant, i.e., the operator T' should satisfy the equation

$$T'(\xi|f_1)T'(\xi|f_2) = T'(\xi|f_1 f_2). \tag{26}$$

We now show that the general solution of eq. (26) is a nonsingular change of the generators

$$T'(\xi|f) = T_\eta(\xi|f) = f(\eta(\xi)),$$

with some functions $\eta^\alpha(\xi)$, $\varepsilon(\eta^\alpha) = 1$.

Let $f_2(\xi) = 1$, then it follows from (26) that

$$T'(\xi|f_1)(1 - T'(\xi|1)) = 0 \quad \Longrightarrow \quad T'(\xi|1) = \int d\xi' T'(\xi|\xi') = 1$$

because T' is assumed to be nonsingular.

We now choose $f_1(\xi) = \exp(\xi^\alpha p_\alpha)$, $f_2(\xi) = \exp(\xi^\alpha q_\alpha)$, where p_α and q_α are odd generators (in the wider Grassman algebra with the generators ξ^α, p^α, q^α), and introduce the notation

$$\int d\xi f(\xi) e^{\xi^\alpha p_\alpha} \equiv \tilde{f}(p).$$

It follows from relation (26) that

$$\tilde{T}'(\xi|p)\tilde{T}'(\xi|q) = \tilde{T}'(\xi|p+q), \quad \tilde{T}'(\xi|0) = 1,$$

whence we obtain

$$\tilde{T}'(\xi|p) = e^{\eta_\alpha(\xi)p_\alpha}, \quad \eta^\alpha(\xi) \equiv \tilde{T}'(\xi|p)\frac{\overleftarrow{\partial}}{\partial p_\alpha}\Big|_{p=0}.$$

The inverse Fourier transform of $\tilde{T}'(\xi|p)$ is

$$T'(\xi|\xi') = \delta(\eta(\xi) - \xi').$$

Thus, we have proved the following.

Theorem

Any associative $*$–product on the Grassman algebra $\mathcal{G}_K$ with boundary condition (5) and the nonsingular symplectic metric in the first order deformation $*_1$ is equivalent to (i.e., is related by a similarity transformation to) the $*$–product $P_G^{\hbar\lambda}(\xi|f_1, f_2)$, which is not equivalent to the "pointwise" product for $\hbar \neq 0$,. We also note that the $*_{G(\hbar\lambda)}$–products with different values of the parameter $\hbar$ can be related by a similarity transformation (generated by a scale transformation of the generators ξ^α).

5 Relation to the Clifford algebra

In this Section, we discuss the relation of the algebra $\mathcal{F}_K$, the algebra of the elements of the Grassman algebra $\mathcal{G}_K$ (i.e., of the functions of the generators) with the $*$–product as a product, to the Clifford algebra $\mathcal{K}_K$ over the field K.

We take the $*_{G(\lambda)}$–product[2] for a product in the algebra $\mathcal{F}_K$. The comment on the general $*$–product is at the end of the Section.

It is easy to verify that the $*_{G(\lambda)}$–product has the following properties:

$$c *_{G(\lambda)} f(\xi) = f *_{G(\lambda)} c(\xi) = cf(\xi) \tag{27}$$

for any function $f(\xi)$ and $c = const$,

$$\xi^{\alpha_1} *_{G(\lambda)} \xi^{\alpha_2} *_{G(\lambda)} \cdots *_{G(\lambda)} \xi^{\alpha_k} = \xi^{\alpha_1}\xi^{\alpha_2}\cdots\xi^{\alpha_k}, \quad \alpha_1 < \alpha_2 < \cdots < \alpha_k. \tag{28}$$

The explicit calculation gives

$$\xi^\alpha *_{G(\lambda)} \xi^\beta = \xi^\alpha\xi^\beta + \lambda_\alpha\delta_{\alpha\beta},$$

therefore, we have

$$\xi^\alpha *_{G(\lambda)} \xi^\beta + \xi^\beta *_{G(\lambda)} \xi^\alpha = 2\lambda_\alpha\delta_{\alpha\beta}, \tag{29}$$

i.e., the generators ξ^α form the Clifford algebra with the $*_{G(\lambda)}$–product.

Collecting together definition (1) and properties (2), (27), (28), and (29), we obtain that the following Statement is true:

The algebra $\mathcal{F}_K$ with the $*_{G(\lambda)}$–product as a product is isomorphic to the Clifford algebra $\mathcal{K}_K$ over K with the generators γ^α, $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\lambda_\alpha\delta_{\alpha\beta}$. The isomorphism $\mathcal{F}_K \Longleftrightarrow \mathcal{K}_K$ is given by

$$\mathcal{F}_K \ni f(\xi) = \sum_I f_I e^I \quad \Longleftrightarrow \quad K_f = \sum_I f_I \Gamma^I \in \mathcal{K}_K, \tag{30}$$

$$f_1 *_{G(\lambda)} f_2(\xi) \quad \Longleftrightarrow \quad K_{f_1}K_{f_2},$$

where

$$\Gamma^0 = 1, \quad \Gamma^\alpha = \gamma^\alpha, \quad \Gamma^{\alpha_1\alpha_2\cdots\alpha_k} = \gamma^{\alpha_1}\gamma^{\alpha_2}\cdots\gamma^{\alpha_k}, \quad k \geq 2.$$

We note that there is the natural Z_2–grading g in this algebra, namely, $g(f)$, where f is considered an element of the algebra $\mathcal{F}_K$, coincides with the Grassman parity $\varepsilon(f)$, where f is considered an element of the algebra $\mathcal{G}_K$. If we introduce the Z_2–grading g in the Clifford algebra,

$$g(\Gamma^I) = k\,(mod\ 2), \quad I = \alpha_1\cdots\alpha_k,$$

then isomorphism (30) preserves the grading.

What we can say about the representations of the algebra $\mathcal{F}_K$, which are simultaneously the representations of the Clifford algebra? It is natural to consider the exact representations.

As is well known, the algebras $\mathcal{K}_{\mathbf{C}}$ and $\mathcal{K}_{\mathbf{R}}$ of even dimension $n = 2m$ and the algebra $\mathcal{K}_{\mathbf{R}}$ of odd dimension $n = 2m+1$ with $\delta\lambda_{(n)} \equiv \sum_\alpha \lambda_\alpha = 3\,(mod\ 4)$[3] have only one irreducible

[2] In this Section, we choose $\hbar = 1$ that is equivalent to the transition to the new generators $\xi'^\alpha = \sqrt{\hbar}\xi^\alpha$

[3] The property $(\Gamma^{12\cdots n})^2 = -1$ holds for this algebra $\mathcal{K}_{\mathbf{R}}$.

representation (for fixed n and $\delta\lambda_{(n)}$), the representation being exact[4]. In these cases, the algebra $\mathcal{F}_K$ has a unique exact irreducible representation coinciding with the irreducible representation of the algebra $\mathcal{K}_K$.

The algebra $\mathcal{K}_{\mathbf{C}}$ of odd dimension $n = 2m+1$ and the algebra $\mathcal{K}_{\mathbf{R}}$ of odd dimension $n = 2m+1$ and $\delta\lambda_{(n)} = 1\,(mod\,4)$[5] are decomposed into a direct sum of two simple subalgebras. One of these subalgebras is realized by zero in the irreducible representations. Therefore the irreducible representations of the Clifford algebra can not be the exact representations of the algebra $\mathcal{F}_K$ (and of the algebra $\mathcal{K}_K$). The minimum exact representation V of the algebra $\mathcal{F}_K$ is decomposed into a direct sum of two irreducible nonequivalent representations, $V = V_+ + V_-$, the projectors $P_\pm$ on $V_\pm$ are $P_\pm = (1 \pm \Gamma^{12\cdots n})/2$.

Finally, we consider the general $*$–product. Let T be the operator generating the similarity transformation of the $*$–product to the $*_{G(\lambda)}$–product:

$$f_1 * f_2 = T^{-1}\Big(Tf_1 *_{G(\lambda)} Tf_2\Big).$$

Then the isomorphism $\mathcal{F}_K \Longleftrightarrow \mathcal{K}_K$ is given by

$$\mathcal{F}_K \ni f(\xi) = \sum_I f_I e^I \quad \Longleftrightarrow \quad K_f = \sum_I f_I T^I_J \Gamma^J \in \mathcal{K}_K, \tag{31}$$

$$f_1 * f_2(\xi) \quad \Longleftrightarrow \quad K_{f_1} K_{f_2},$$

where the matrix T^I_J is defined by

$$Te^I = T^I_J e^J.$$

Isomorphism (31) of the algebras $\mathcal{G}_K$ and $\mathcal{K}_K$ preserves the above–introduced grading because we consider only even operators T.

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Appendix

For completeness, we prove the statement that $*_{G(\lambda)}$–product (22) with arbitrary λ_α ($\hbar = 1$) satisfies associativity equation (3). In what follows, we omit the bottom index "$G(\lambda)$" and write the top index "(n)" at the $*$–product symbols, where n is the number of the generators of the Grassman algebra. Similarly, we also omit the bottom index "λ" and write the top index "(n)" at the operator Q in the definition of the $*_G$–product. The operator $Q^{(n)}$ satisfies the relations

$$Q^{(n)} = Q^{(n-1)} + \lambda_n \overleftarrow{\partial}_n \partial_n,$$

$$e^{Q^{(n)}} = (1 + \lambda_n \overleftarrow{\partial}_n \partial_n) e^{Q^{(n-1)}}.$$

We prove the statement by induction in the number n of the generators.

[4]The matrices realizing the basis Γ^I are linearly independent.

[5]The property $(\Gamma^{12\cdots n})^2 = 1$ holds for this algebra $\mathcal{K}_{\mathbf{R}}$.

Let $\varphi(\xi)$ and $\bar\varphi(\xi)$ be the elements of the Grassman algebra that are independent of the generator ξ^n:

$$\frac{\partial}{\partial\xi^n}\varphi(\xi)\equiv 0,\quad \frac{\partial}{\partial\xi^n}\bar\varphi(\xi)\equiv 0.$$

Any element $f(\xi)$ can be represented as

$$f(\xi)=\varphi(\xi)+\xi^n\bar\varphi(\xi),\quad \varepsilon(\varphi)=\varepsilon(f),\quad \varepsilon(\bar\varphi)=\varepsilon(f)+1\,(mod\ 2).$$

The $*^{(n)}$–product is represented in terms of φ and $\bar\varphi$ as follows:

$$f_1*^{(n)}f_2(\xi)=\varphi_{12}(\xi)+\xi^n\bar\varphi_{12}(\xi),$$

$$\begin{aligned}\varphi_{12}(\xi)&=\varphi_1*^{(n-1)}\varphi_2(\xi)+\lambda_n(-1)^{\varepsilon(\bar\varphi_1)}\bar\varphi_1*^{(n-1)}\bar\varphi_2(\xi),\\ \bar\varphi_{12}(\xi)&=\bar\varphi_1*^{(n-1)}\varphi_2(\xi)+(-1)^{\varepsilon(\varphi_1)}\varphi_1*^{(n-1)}\bar\varphi_2(\xi).\end{aligned}$$

The associator of the $*^{(n)}$–product is

$$(f_1*^{(n)}f_2)*^{(n)}f_3(\xi)-f_1*^{(n)}(f_2*^{(n)}f_3)(\xi)=\varphi(\xi)+\xi^n\bar\varphi(\xi),$$

$$\begin{aligned}\varphi=\varphi_{(12)3}-\varphi_{1(23)}=&[(\varphi_1*^{(n-1)}\varphi_2)*^{(n-1)}\varphi_3-\varphi_1*^{(n-1)}(\varphi_2*^{(n-1)}\varphi_3)]+\\ &+\lambda_n(-1)^{\varepsilon(\bar\varphi_1)}[(\bar\varphi_1*^{(n-1)}\bar\varphi_2)*^{(n-1)}\varphi_3-\bar\varphi_1*^{(n-1)}(\bar\varphi_2*^{(n-1)}\varphi_3)]+\\ &+\lambda_n(-1)^{\varepsilon(\bar\varphi_1)+\varepsilon(\varphi_2)}[(\bar\varphi_1*^{(n-1)}\varphi_2)*^{(n-1)}\bar\varphi_3-\bar\varphi_1*^{(n-1)}(\varphi_2*^{(n-1)}\bar\varphi_3)]+\\ &+\lambda_n(-1)^{\varepsilon(\bar\varphi_2)}[(\varphi_1*^{(n-1)}\bar\varphi_2)*^{(n-1)}\bar\varphi_3-\varphi_1*^{(n-1)}(\bar\varphi_2*^{(n-1)}\bar\varphi_3)]\ ,\end{aligned}$$

$$\begin{aligned}\bar\varphi=\bar\varphi_{(12)3}-\bar\varphi_{1(23)}=&[(\bar\varphi_1*^{(n-1)}\varphi_2)*^{(n-1)}\varphi_3-\bar\varphi_1*^{(n-1)}(\varphi_2*^{(n-1)}\varphi_3)]+\\ &+(-1)^{\varepsilon(\varphi_1)}[(\varphi_1*^{(n-1)}\bar\varphi_2)*^{(n-1)}\varphi_3-\varphi_1*^{(n-1)}(\bar\varphi_2*^{(n-1)}\varphi_3)]+\\ &+(-1)^{\varepsilon(\varphi_1)+\varepsilon(\varphi_2)}[(\varphi_1*^{(n-1)}\varphi_2)*^{(n-1)}\bar\varphi_3-\varphi_1*^{(n-1)}(\varphi_2*^{(n-1)}\bar\varphi_3)]+\\ &+\lambda_n(-1)^{\varepsilon(\bar\varphi_2)}[(\bar\varphi_1*^{(n-1)}\bar\varphi_2)*^{(n-1)}\bar\varphi_3-\bar\varphi_1*^{(n-1)}(\bar\varphi_2*^{(n-1)}\bar\varphi_3)]\ .\end{aligned}$$

We can see from these formulas that if the $*^{(n-1)}$–product is associative, then the $*^{(n)}$–product is also associative. In addition, it is easy to see from the same formulas that the $*^{(1)}$–product is associative (all φ_i and $\bar\varphi_i$ are numbers, $\varepsilon(\varphi_i)=\varepsilon(\bar\varphi_i)=0$ and the $*^{(0)}$–product is the usual product of numbers). □

References

[1] M.V.Karasev, V.P.Maslov, Nonlinear Poisson brackets. Geometry and quantization, Moscow, Science, 1991.

[2] Flato V., Sternheimer D., Lett. Math. Phys., 1991, **22**, 155.

[3] Bonneau P., Flato V., Pinczon G., Lett. Math. Phys., 1992, **25**, 75.

[4] N.Nekrasov, Trieste Lectures on solitons in noncommutative gauge theories, hep–th/0011095.

[5] A.Schwarz, Gauge theories on noncommutative spaces, hep–th/0011261.

[6] Fedosov B., Deformation quantization and Index Theory, Akademie Verlag, Berlin, 1996.

[7] Kontsevich Maxim, Deformation quantization of Poisson manifolds, I, q–alg/9709040.

[8] Grigoriev M., Lyakhovich S., Fedosov deformation quantization as BRST theory, hep–th/0003114.

[9] Fedosov B., J. Diff. Geom., 1994, **40**, 213.

[10] Groenewold H., Physica, 1946, **12**, 405 - 460.

[11] Berezin F.A., Sov. Phys. Usp., 1980, **23**, 1981.

[12] Batalin I.A., Fradkin E.S., Riv. Nuovo Cim., 1986, **9**, 1.

[13] Fradkin E., Linetsky V., Mod. Phys. Lett. A, 1991, **6**, 217.

[14] Gozzi E., Reuter M., Mod. Phys. Lett. A, 1993, **8**, 1433.

[15] Batalin I.A., Tyutin I.V., J. Math. Phys., 1993, **34**, 369.